A CHARACTERIZATION OF IRREDUCIBLE HERMITIAN SYMMETRIC SPACES OF TUBE TYPE BY $\mathbb{C}^*$-ACTIONS

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ABSTRACT. A $\mathbb{C}^*$-action on a projective variety X is said to be of Euler type at a nonsingular fixed point x if the isotropy action of $\mathbb{C}^*$ on T_xX is by scalar multiplication. In this paper, it is proven that a smooth projective variety of Picard number one X is isomorphic to an irreducible Hermitian symmetric space of tube type if and only if for a general pair of points x, y on X, there exists a $\mathbb{C}^*$-action on X which is of Euler type at x and its inverse action is of Euler type at y.

1. INTRODUCTION

1.1. **Main result.** The study of complex torus actions on algebraic varieties is a classical topic in algebraic geometry. It is an interesting problem to classify algebraic varieties with special $\mathbb{C}^*$-actions. Let X be a smooth projective variety. A $\mathbb{C}^*$-action on X is said to be equalized at a fixed point x if any weight of the isotropy action on the tangent space T_xX equals to 0 or ± 1. We call the action to be equalized if it is equalized at each fixed point. Denote by $X^{\mathbb{C}^*}$ the set of fixed points of the $\mathbb{C}^*$-action. An irreducible component of $X^{\mathbb{C}^*}$ is called extremal if it intersects with general $\mathbb{C}^*$-orbit closures. In the series of works [RW20] [ORSW22] [ORSW21], the authors study equalized $\mathbb{C}^*$-actions on projective manifolds with isolated extremal fixed components. For rational homogenous spaces they proved the following:

Theorem 1.1. *[ORSW21] Let $X = G/P$ be a rational homogeneous space of Picard number one, then X admits an equalized $\mathbb{C}^*$-action with two isolated extremal fixed points if and only if X is isomorphic to one of the followings:*
(i) a smooth hyperquadric $\mathbb{Q}^n$,
(ii) the Grassmaninan variety $Gr(n, 2n)$,
(iii) the Lagrangian Grassmannian variety $Lag(n, 2n)$,
(iv) the spinor variety $\mathbb{S}_{2n}$,
(v) the 27 dimensional E_7-variety E_7/P_7.

The varieties classified above are exactly irreducible Hermitian symmetric spaces (IHSS for short) of tube type. Recall that an IHSS is said to be of tube type if its dual, as a bounded symmetric domain, is holomorphically equivalent to a tube domain over a self-dual cone. It is therefore a natural problem to characterize IHSS of tube type by $\mathbb{C}^*$-actions in a more general context. A $\mathbb{C}^*$-action on a projective manifold X is said to be of Euler type at a fixed point x if the isotropy action of $\mathbb{C}^*$ on T_xX is by scalar multiplication. In Theorem 1.1, by taking certain conjugates of $\mathbb{C}^*$ in the automorphism group of X one can show that for a general pair of points x, y on X there exists a $\mathbb{C}^*$-action which is of Euler type at x and its inverse action is of Euler type at y. Here, a general pair of points (x, y) in $X \times X$ is one that lies in some Zariski open dense subset. Our main result proves the converse:

Theorem 1.2. *Let X be a smooth projective variety of Picard number one, then X is isomorphic to an IHSS of tube type if and only if for a general pair of points x, y on X, there exists a $\mathbb{C}^*$-action on X which is of Euler type at x and its inverse action is of Euler type at y.*

1.2. **Outline of the proof.** The main ingredient of the proof is the theory of varieties of minimal rational tangents (VMRT for short) developed by Hwang and Mok. Let X be a Fano manifold of Picard number one and let $\mathcal{K}$ be a fixed irreducible dominant family of minimal rational curves on X. The VMRT at a general point $x \in X$ is the closed subvariety $\mathcal{C}_x \subseteq \mathbb{P}T_xX$ consisting of the closure of tangent directions at x of general curves in $\mathcal{K}$ passing through x. A large part of the global geometry of the manifold is encoded in the VMRT $\mathcal{C}_x \subseteq \mathbb{P}T_xX$ at a general point x.. The first step is to show that X

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in Theorem 1.2 is an equivariant compactification of vector group, hence X can be recovered from its VMRT by Cartan-Fubini type theorem [HM01, Theorem 1.2]. Then we follow the methods developed in [HM05] [FH12] to classify the projective subvariety $\mathcal{C}_x \subseteq \mathbb{P}T_xX$ by studying the prolongation of its infinitesimal linear automorphisms.

A crucial step of our proof is the following result.

Theorem 1.3. *Let X be a smooth projective variety of Picard number one. Assume for a general pair of points $(x,y) \in X \times X$, there is a $\mathbb{C}^*$-action on X which is of Euler type at x and its inverse action is of Euler type at y, then for the VMRT $\mathcal{C}_x \subseteq \mathbb{P}T_xX$ we have:*

$$dim(\mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)}) = dim(X) \tag{1.1}$$

Here, $\mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)}$ denotes the space of first prolongations of the VMRT $\mathcal{C}_x \subset \mathbb{P}T_xX$. We refer to Section 2.2 for the precise definitions of the VMRT and its prolongations.

The identity (1.1) is known to hold for any IHSS (see Section 2.3). Now assume that X satisfies the conditions of Theorem 1.2. The natural actions of the vector groups T_yX and T_xX on themselves can be extended to equivariant compactifications on X. Choosing a suitable non-degenerate projective embedding $X \subseteq \mathbb{P}V$. The actions of T_yX and T_xX on X can be lifted to linear actions on V (Theorem 3.1). Identifying $Lie(T_yX), Lie(T_xX)$ with their images in $\mathfrak{gl}(V)$, the adjoint actions of the two Lie subalgebras inside $\mathfrak{gl}(V)$ will induce the identification: $Lie(T_yX) \cong \mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)}$ (Proposition 4.5).

In [FH12] Fu and Hwang classified irreducible non-degenerate nonsingular projective subvariety $S \subseteq \mathbb{P}V$ with nonzero prolongations. Most of them are the VMRT of an IHSS or a nonsingular linear section of some IHSS. Using a case-by-case calculation of $\dim(\mathfrak{aut}(\hat{S})^{(1)})$, we show that $\mathcal{C}_x \subseteq \mathbb{P}T_xX$ is projectively isomorphic to the VMRT of an IHSS (Proposition 4.8). Then, by a $\mathbb{C}^*$-equivariant Cartan-Fubini extension theorem (Corollary 5.1), we conclude that X is $\mathbb{C}^*$-equivariantly isomorphic to an IHSS. The $\mathbb{C}^*$-action on IHSS was studied in detail in [ORSW21], as a corollary it has two isolated extremal fixed points if and only if it is isomorphic to an IHSS of tube type.

The article is organized as follows. In Section 2 we recall basic definitions and facts used in the sequel, illustrating them in the context of IHSS. In Section 3 we first review the approach in [FH20] to associate $\mathbb{C}^*$-actions with vector group actions, then apply it to the case when the $\mathbb{C}^*$-action has two isolated extremal fixed points. Section 4 studies prolongations of projective varieties. We first prove Theorem 1.3, then calculate the dimension of prolongation for certain projective subvarieties with nonzero prolongations. Section 5 completes the proof of our main result.

Notations. Throughout this article we work over the field of complex numbers. Given a line bundle $\mathcal{L}$ on a variety X, the principal open subset of a section $s \in H^0(X,\mathcal{L})$ is denoted by $D_+(s) = \{x \in X : s(x) \neq 0\}$, and the cycle-theoretic zero locus of s is denoted by $Z(s)$. For a vector space V of dimension n, $\mathbb{C}[V] \cong \oplus_{k\geqslant 0} \operatorname{Sym}^k(V^*)$ is realized by assigning a function f on V the polynomial $P_f = \sum_{k\geqslant 0} P_{f,k}$ such that $f(v) = \sum_{k\geqslant 0} P_{f,k}(v,v,\dots,v)$.

2. Preliminaries

2.1. **$\mathbb{C}^*$-actions, vector group actions, and Euler-symmetric varieties.** In this section we introduce some notations and conventions on $\mathbb{C}^*$-actions and vector group actions.

Definition 2.1. *Let X be a projective variety with a $\mathbb{C}^*$-action.*

(1) *For a point $x \in X$, the source x_+ and the sink x_- of x are defined as*

$$x_+ = \lim_{t\to 0} t\cdot x, \quad x_- = \lim_{t\to\infty} t\cdot x.$$

(2) *Denote by $X^{\mathbb{C}^*}$ the fixed locus of the action, and by $\mathcal{Y}$ the set of irreducible components of $X^{\mathbb{C}^*}$. The sink X_0 and source X_∞ are the extremal components characterized by the property that for a general point $x \in X$, we have $x_+ \in X_0$ and $x_- \in X_\infty$.*

(3) *When X is smooth, each fixed component $Y \in \mathcal{Y}$ is smooth by [Ive72]. For each $Y \in \mathcal{Y}$, the Bialynicki-Birula cells are defined as*

$$C^{\pm}(Y) = \{x \in X \mid x_\pm \in Y\}.$$

The isotropy action of $\mathbb{C}^*$ on $TX|_Y$ induces a decomposition

$$TX|_Y = T^+(Y) \oplus T^-(Y) \oplus TY,$$

where $T^{+}(Y)$ and $T^{-}(Y)$ are the subbundles on which $\mathbb{C}^*$ acts with positive and negative weights, respectively.

We recall the following theorem of Białynicki-Birula [BB73].

Theorem 2.2. *Let X be a smooth projective variety with a $\mathbb{C}^*$-action.*

(1) *For each $Y \in \mathcal{Y}$, the sets $C^{\pm}(Y)$ are locally closed, and there are decompositions*

$$X = \bigcup_{Y \in \mathcal{Y}} C^{+}(Y) = \bigcup_{Y \in \mathcal{Y}} C^{-}(Y).$$

(2) *For each $Y \in \mathcal{Y}$, there are $\mathbb{C}^*$-equivariant isomorphisms $C^{+}(Y) \cong T^{+}(Y)$ and $C^{-}(Y) \cong T^{-}(Y)$, lifting the natural maps $C^{\pm}(Y) \to Y$. Moreover, these maps are algebraic and give $\mathbb{C}^{v^{\pm}(Y)}$-fibrations, where $v^{\pm}(Y) = \operatorname{rank}(T^{\pm}(Y))$.*

For a $\mathbb{C}^*$-action on a normal projective variety X equipped with a very ample line bundle $\mathcal{L}$, we define its *normalized linearization* as follows.

Lemma 2.3. *Let X be a normal projective variety with a $\mathbb{C}^*$-action, and $\mathcal{L}$ a very ample line bundle on X.*

(1) *There exists a linearization of the $\mathbb{C}^*$-action on $\mathcal{L}$ (cf. [Bri15, Theorem 2.14]). This induces an action on $H^0(X, \mathcal{L})$ with a weight decomposition:*

$$H^0(X, \mathcal{L}) = \bigoplus_{k=0}^{r} H^0(X, \mathcal{L})_{w_k},$$

where each $H^0(X, \mathcal{L})_{w_k}$ is a nonzero weight subspace of weight w_k.

(2) *The linearization is normalized if the weights are ordered as*

$$0 = w_0 > w_1 > \cdots > w_r.$$

Such a normalization is unique, since any two linearizations differ by a character of $\mathbb{C}^*$.

We recall the definition of an equivariant compactification of vector group.

Definition 2.4. *Let $G = \mathbb{C}^n$ be the complex vector group of dimension n. An equivariant compactification of the vector group $G \cong \mathbb{C}^n$ is a projective variety X of dimension n equipped with an algebraic action of G that has a Zariski open orbit $O \subset X$. In particular the orbit O is G-equivariantly isomorphic to G.*

Next, we introduce *Euler-symmetric varieties*, as defined in [FH20]. These varieties arise as equivariant compactifications of vector groups and are characterized by the presence of Euler type $\mathbb{C}^*$-actions.

Definition 2.5. *Let $Z \subseteq \mathbb{P}V$ be a projective subvariety. A $\mathbb{C}^*$-action on Z is called of Euler type at a nonsingular fixed point x if the isotropy action on the tangent space T_xZ is by scalar multiplication. We say $Z \subseteq \mathbb{P}V$ is an Euler-symmetric variety if for a general point $x \in Z$, there exists a $\mathbb{C}^*$-action which is of Euler type at x, where the $\mathbb{C}^*$-action comes from a multiplicative subgroup of $GL(V)$.*

While our definition differs slightly from that of [FH20], the equivalence between these formulations follows from the fundamental characterizations of Euler-symmetric varieties given below.

Theorem 2.6. *Let X be a normal projective variety equipped with a very ample line bundle $\mathcal{L}$. The following conditions are equivalent:*

(1) *For a general point $x \in X$, there exists a $\mathbb{C}^*$-action on X that is of Euler type at x;*
(2) *X is an equivariant compactification of a vector group, and the scalar multiplication of $\mathbb{C}^*$ on this vector group extends to a $\mathbb{C}^*$-action on X;*
(3) *The projective embedding $X \subset \mathbb{P}H^0(X, \mathcal{L})^{\vee}$ is Euler-symmetric.*

Proof. (3) $\Rightarrow$ (2) follows from [FH20, Theorem 3.7]. The implication (2) $\Rightarrow$ (1) is clear. Finally, (1) $\Rightarrow$ (3) follows from Lemma 2.3(1). ☐

2.2. **VMRT and prolongations.** The main ingredients of our study are the VMRT theory of Fano manifolds and prolongations of projective subvarieties. Below we introduce the basic definitions and results used in what follows.

Definition 2.7. *Let X be a Fano manifold of Picard numbr one. An irreducible component $\mathcal{K}$ of the space* $\mathrm{Ratcurves}^n(X)$ *of rational curves on X is called a minimal rational component if the subvariety $\mathcal{K}_x$ of $\mathcal{K}$ parameterizing curves passing through a general point $x \in X$ is non-empty and proper. Curves parameterized by $\mathcal{K}$ are called minimal rational curves. Let $\rho : \mathcal{U} \to \mathcal{K}$ be the universal family and $\mu : \mathcal{U} \to X$ the evaluation map. The tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$ is defined by $\tau(u) = T_{\mu(u)}(\mu(\rho^{-1}(\rho(u))))$. The closure $\mathcal{C} \subset \mathbb{P}T(X)$ of its image is the total space of variety of minimal rational tangents. The natural projection $\mathcal{C} \to X$ is a proper surjective morphism and a general fiber $\mathcal{C}_x \subset \mathbb{P}T_xX$ is called the variety of minimal rational tangents (VMRT for short) at the point $x \in X$.*

Let us recall the Cartan-Fubini type extension theorem proved by Hwang and Mok [HM01]. We will use the following version taken from [FH12, Theorem 6.8]

Theorem 2.8. *Let X_1 and X_2 be two Fano manifolds of Picard number 1, different from projective spaces. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be families of minimal rational curves on X_1 and X_2 respectively. Assume that for a general point $x \in X_1$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X_1)$ is irreducible and nonsingular. Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be connected analytical open subsets. Suppose that there exists a biholomorphic map $\psi : U_1 \to U_2$ such that for a general point $x \in U_1$, the differential $d\psi_x : \mathbb{P}T_x(U_1) \to \mathbb{P}T_{\psi(x)}(U_2)$ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}_{\psi(x)}$. Then there exists a biregular morphism $\Psi : X_1 \to X_2$ such that $\psi = \Psi|_{U_1}$.*

Remark 2.9. Let X be a smooth projective variety of Picard number one satisfying one of the conditions in Theorem 2.6. Then:

(1) X is a Fano manifold because it is uniruled;
(2) For a suitably chosen family of minimal rational curves $\mathcal{K}$, the associated VMRT is irreducible and nonsingular at general points (Proposition 4.2).

Definition 2.10. (1) *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra. The k-th prolongation of $\mathfrak{g}$ is the space of symmetric multilinear homomorphisms $A : \mathrm{Sym}^{k+1}V \to V$ such that for any fixed $v_1, \ldots, v_k \in V$, the linear map $A_{v_1,\ldots,v_k} : V \to V$ defined by*

$$v \mapsto A(v, v_1, \ldots, v_k)$$

belongs to $\mathfrak{g}$.

(2) *Let $S \subseteq \mathbb{P}V$ be a projective subvariety with affine cone $\hat{S} \subseteq V$. The Lie algebra of infinitesimal linear automorphisms of $\hat{S}$ is*

$$\mathfrak{aut}(\hat{S}) = \left\{ g \in \mathrm{End}(V) \;\middle|\; \begin{array}{l} g(\alpha) \in T_\alpha(\hat{S}) \text{ for all smooth points } \alpha \in \hat{S} \\ \text{equivalently, } \exp(tg) \cdot \hat{S} \subset \hat{S} \text{ for all } t \in \mathbb{C} \end{array} \right\}.$$

Its k-th prolongation $\mathfrak{aut}(\hat{S})^{(k)}$ is called the k-th prolongation of $S \subseteq \mathbb{P}V$.

We now state some fundamental results about prolongations:

Theorem 2.11. *[HM05, Theorems 1.1.2 and 1.1.3] Let $Y \subset \mathbb{P}V$ be an irreducible, smooth, non-degenerate, and linearly normal subvariety.*

(1) $\mathfrak{aut}(\hat{Y})^{(2)} = 0$.
(2) *For any non-zero $A \in \mathfrak{aut}(\hat{Y})^{(1)}$, there exists a unique non-zero linear functional $\lambda_A \in V^*$ satisfying:*
   (a) *For all $\alpha \in V$,*
   $$A_{\alpha,\alpha} = \lambda_A(\alpha)\alpha;$$
   (b) *For all $\alpha \in \hat{Y}$ and $\alpha' \in T_\alpha(\hat{Y})$,*
   $$\lambda_A(\alpha)\alpha' + \lambda_A(\alpha')\alpha = 2A_{\alpha,\alpha'}.$$

Consequently, for any point $y = [\alpha] \in Y$ with $\lambda_A(\alpha) \neq 0$, there exists $E_y \in \mathfrak{aut}(\hat{Y})$ that generates a $\mathbb{C}^*$-action on Y of Euler type at y.

Furthermore, in [FH12,FH18] Fu and Hwang provided a complete classification of irreducible, smooth and non-degenerate projective subvariety with non-zero prolongations. See Theorem 4.6 for a full list.

2.3. **Examples: irreducible Hermitian symmetric spaces.** In this subsection we illustrate our definitions from previous sections in the context of rational homogeneous spaces of Picard number one. Most of this material can be found in [Arz11], [FH20], [HM05], and [ORSW21].

2.3.1. *G/P as Euler-symmetric varieties.* Let G be a simple algebraic group of adjoint type. Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus. We denote by Φ the root system, $\Delta \subset \Phi$ the set of simple roots, and $\mathcal{D}$ the Dynkin diagram. Let $\mathfrak{g} = \mathrm{Lie(G)}$ be the Lie algebra of G with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. A rational homogeneous G-variety is a projective variety of the form $X = G/P$. Up to conjugacy, it can be obtained as follows. For any subset $I \subset \Delta$, we denote by $P_I = BW_{\mathcal{D}\backslash I}B$ the standard parabolic subgroup associated to I, where $W_{\mathcal{D}\backslash I}$ is the subgroup of the Weyl group W generated by simple reflections corresponding to $\mathcal{D}\backslash I$. Similarly, let P_I^- be the opposite parabolic subgroup. The associated rational homogeneous space is denoted by $X = G/P_I$, which is a smooth projective rational variety of Picard number $|I|$. In what follows, we always assume $I = \{\alpha\}$ for some simple root $\alpha \in \Delta$. We denote by $H \subset P_\alpha$ the Levi subgroup.

For any root $\beta \in \Phi$, let $m_\alpha(\beta)$ denote the multiplicity of α in β. This induces a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k, \quad \mathfrak{g}_k = \bigoplus_{m_\alpha(\beta)=k} \mathfrak{g}_\beta.$$

Proposition 2.12. *[Arz11] [FH20] Let $X = G/P_\alpha$ be a rational homogeneous space of Picard number one. Assume $G \cong \mathrm{Aut}^0(X)$ is of adjoint type. Then the following conditions are equivalent:*

1. *$X \subset \mathbb{P}V$ is an Euler-symmetric variety under its minimal G-equivariant embedding given by the generator of $\mathrm{Pic}(X)$;*
2. *X is an equivariant compactification of a vector group;*
3. *The grading of $\mathfrak{g}$ with respect to α is short, i.e., $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.*

In case (3), we have $\mathfrak{g}_0 = \mathrm{Lie(H)}$, $\mathfrak{g}_1 = \mathrm{Lie(R_u(P_\alpha))}$, and $\mathfrak{g}_{-1} = \mathrm{Lie(R_u(P_\alpha^-))}$.

Remark 2.13. $X = G/P \subset \mathbb{P}V$ is called an irreducible Hermitian symmetric space (IHSS) if it satisfies any of the above equivalent conditions. In this case, we can describe the Euler-type $\mathbb{C}^*$-action and the vector group action on X as follows. Denote by $x = [P_\alpha], y = w_0 \cdot x$, where $w_0 \in W$ is the longest element in the Weyl group.

1. Both $R_u(P_\alpha^-)$ and $R_u(P_\alpha)$ are vector groups, and X is an equivariant compactification of $R_u(P_\alpha^-)$ with the open orbit $R_u(P_\alpha^-) \cdot x$;
2. There exists a unique element $H \in \mathfrak{t}$ satisfying $\beta(H) = -\delta_{\alpha,\beta}$ for all $\beta \in \Delta$. The corresponding one-parameter subgroup induces a $\mathbb{C}^*$-action on X that is of Euler type at x.
3. The $\mathbb{C}^*$-action in (2) has an isolated source if and only if X is isomorphic to an IHSS of tube type (as classified in Theorem 1.1), where the source is given by $y = w_0 \cdot x$. In this case, the inverse action is of Euler type at y, and X is an equivariant compactification of the vector group $R_u(P_\alpha)$, with the open orbit being $R_u(P_\alpha) \cdot y$.

2.3.2. *VMRT of an IHSS and its prolongations.* Maintaining the above notation, assume $X = G/P_\alpha \subset \mathbb{P}V$ is an IHSS under its minimal embedding as in Proposition 2.12. Then X is covered by a unique family $\mathcal{K}$ of lines in $\mathbb{P}V$. Denote by $\mathcal{C} \subset \mathbb{P}T(X)$ the associated VMRT structure as in Definition 2.7. Fix $x = [P_\alpha]$ as the base point which is fixed by H. We now describe the VMRT at x and its prolongations.

1. $T_xX \cong \mathfrak{g}_1$, and $\mathcal{C}_x \subset \mathbb{P}T_xX$ is the unique closed H-orbit in $\mathbb{P}\mathfrak{g}_1$;
2. $\mathfrak{aut}(\hat{\mathcal{C}}_x) \cong \mathfrak{g}_0$ via the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$;
3. $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \cong \mathfrak{g}_{-1}$ via the adjoint action:

$$\mathfrak{g}_{-1} \to \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_0), \quad \alpha \mapsto (\beta \mapsto [\alpha, \beta]), \tag{2.1}$$

where the image lies in $\mathrm{Sym}^2 T_x^*X \otimes T_xX$ since $\mathfrak{g}_1$ is abelian.

We list IHSS and their VMRTs in the following table.

TABLE 1. IHSS and their VMRTs

IHSS $X = G/P$	$\mathbb{Q}^n$	$Gr(a, a+b)$	$\mathbb{S}_n$	$Lag(n, 2n)$	E_6/P_1	E_7/P_7
VMRT $\mathcal{C}_x$	$\mathbb{Q}^{n-2}$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	$Gr(2, n)$	$\mathbb{P}^{n-1}$	$\mathbb{S}_5$	E_6/P_1
$\mathcal{C}_x \subset \mathbb{P}T_x(X)$	Hyperquadric	Segre	Plücker	second Veronese	Spinor	Severi

3. From $\mathbb{C}^*$-action to vector group action

The main result in this section is to refine the characterizations of Euler-symmetric varieties given in Theorem 2.6, adapting them in our setting.

Theorem 3.1. *(Proposition 3.7 and Corollary 3.8) Let X be a smooth projective variety of Picard number one. Suppose that for a general pair of points $(x, y) \in X \times X$, there exists a $\mathbb{C}^*$-action on X which is of Euler type at x, and its inverse action is of Euler type at y.*

1. *Fix $\mathbb{C}^*$-equivariant isomorphisms $C^+(x) \cong T_xX$ and $C^-(y) \cong T_yX$ as in Theorem 2.2(2). There exist unique vector group actions of T_xX and T_yX on X, extending their actions on $C^+(x)$ and $C^-(y)$, respectively.*
2. *Let $\mathcal{L}$ be a very ample line bundle on X, and let $V = H^0(X, \mathcal{L})^\vee$ with $f\colon X \hookrightarrow \mathbb{P}V$ the corresponding projective embedding. Then the normalized $\mathbb{C}^*$-linearization (Lemma 2.3) induces a weight decomposition*
$$V = \bigoplus_{k=0}^{r} V_k,$$
where each V_k has weight k, with V_0 and V_r one-dimensional subspaces satisfying $f(x) = [V_0]$ and $f(y) = [V_r]$.
Moreover, the actions of T_xX and T_yX on X uniquely lift to V via representations
$$\rho_x\colon T_xX \to \mathrm{GL}(V), \quad \rho_y\colon T_yX \to \mathrm{GL}(V),$$
satisfying the weight compatibility conditions:
$$d\rho_x(V_k) \subset V_{k+1}, \quad d\rho_y(V_k) \subset V_{k-1}.$$

The vector group actions are defined in terms of the fundamental forms, as established in [FH20, Theorem 3.7] (see also (3.3)). We first recall the definition of fundamental forms, following the conventions of [LM03] and [FH20].

Definition 3.2. *(1) Let $x \in X \subseteq \mathbb{P}V$ be a nonsingular point of a non-degenerate projective variety, and let $\mathcal{L} = \mathcal{O}_{\mathbb{P}V}(1)|_X$ be the line bundle on X. For each nonnegative integer k, $\mathfrak{m}_{x,X}^k$ be the k-th power of the maximal ideal $\mathfrak{m}_{x,X}$. For a section $s \in H^0(X, \mathcal{L})$, let $j_x^k(s)$ be the k-jet of s at x such that $j_x^0 = s_x \in \mathcal{L}_x$. The induced homomorphism :*
$$(V^\vee \cap Ker(j_x^{k-1}))/(V^\vee \cap Ker(j_x^k)) \to \mathcal{L}_x \otimes Sym^k T_x^* X$$
is injective. For each $k \geqslant 2$, the subspace $\mathbb{F}_x^k \subseteq Sym^k T_x^*X$ defined by the image of this homomorphism is called the k-th fundamental form of X at x. Set $F_x^0 = \mathbb{C}$ and $F_x^1 = T_x^*X$. The collection of subspaces $\mathbb{F}_x = \oplus_{k\geqslant 0}\mathbb{F}_x^k \subset \oplus_{k\geqslant 0} Sym^k T_x^* X$ is called the system of fundamental forms of X at x.

(2) Let W be a vector space. For $w \in W$, the contraction homomorphism $\iota_w : Sym^{k+1}W^\vee \to Sym^k W^\vee$ sending $\phi \in Sym^{k+1}W^\vee$ to $\iota_w\phi \in Sym^k W^\vee$ is defined by:
$$\iota_w\phi(w_1, ..., w_k) = \phi(w, w_1, ..., w_k)$$
for any $w_1, ..., w_k \in W$. By convention we define $\iota_w(Sym^0 W^\vee) = 0$.

(3) A subspace $\mathbb{F} = \oplus_{k\geqslant 0}\mathbb{F}^k \subset \oplus_{k\geqslant 0} Sym^k W^\vee$ with $\mathbb{F}^0 = \mathbb{C}, \mathbb{F}^1 = W^\vee, \mathbb{F}^r \neq 0$, and $\mathbb{F}^{r+i} = 0$ for all $i \geqslant 1$ is called a symbol system of rank r if $\iota_w \mathbb{F}^{k+1} \subseteq \mathbb{F}^k$ for any $w \in W$ and any $k \geqslant 0$.

We recall the following theorem of Cartan (see for example [LM03, Section 2.1]).

Theorem 3.3. *Let $X \subseteq \mathbb{P}V$ be a non-degenerate projective subvariety, and let $x \in X$ be a general point. Then the system of fundamental forms $\mathbb{F}_x \subset \oplus_{k\geqslant 0} Sym^k T_x^* X$ is a symbol system.*

In the rest of this section, we adopt the following assumption.

Definition 3.4. *Let X be a smooth projective variety admitting a $\mathbb{C}^*$-action of Euler type at a fixed point x. Let $\mathcal{L}$ be a very ample line bundle on X, and set $V = H^0(X, \mathcal{L})^\vee$. Fix the normalized $\mathbb{C}^*$-linearization of $\mathcal{L}$, and consider the weight space decomposition*
$$H^0(X, \mathcal{L}) = \bigoplus_{k=0}^{r} H^0(X, \mathcal{L})_{w_k},$$
where the weights satisfy
$$0 = w_0 > w_1 > w_2 > \cdots > w_r.$$

We denote by $V_k = H^0(X,\mathcal{L})^\vee_{w_k}$ the subspace of V of weight $-w_k$. We further fix a $\mathbb{C}^*$-equivariant isomorphism $C^+(x) \cong T_xX$ for which the differential at the origin (the point x) is the identity map.

Then the fundamental forms of $X \subset \mathbb{P}V$ at x can be expressed as follows:

Proposition 3.5. *In the setting of Definition 3.4, we have:*

(1) *The subspace*
$$\{s \in H^0(X,\mathcal{L}) \mid D_+(s) = C^+(x)\}$$
is one-dimensional and equals $H^0(X,\mathcal{L})_0$. Take a nonzero section s_0 with $D_+(s_0) = C^+(x)$ and consider the linear map:
$$\eta\colon H^0(X,\mathcal{L}) \to \mathbb{C}[C^+(x)], \qquad s \mapsto \eta(s),$$
defined by $s|_{C^+(x)} = \eta(s)s_0|_{C^+(x)}$ for any $s \in H^0(X,\mathcal{L})$. Then:

(2) *The map η is an injective $\mathbb{C}^*$-equivariant linear map, where the $\mathbb{C}^*$-action on $\mathbb{C}[C^+(x)]$ is given by*
$$(z\cdot f)(u) = f(z^{-1}\cdot u)$$
for any $z \in \mathbb{C}^*$, $u \in C^+(x)$, and $f \in \mathbb{C}[C^+(x)]$.

(3) *Under the identification*
$$\mathbb{C}[C^+(x)] \cong \mathbb{C}[T_xX] \cong \bigoplus_{k\geq 0}\mathrm{Sym}^k T_x^*X,$$
the image of η is $\mathbb{F}_x$, where $\eta(H^0(X,\mathcal{L})_{w_k}) = \mathbb{F}_x^{-w_k}$. Furthermore, $w_1 = -1$, and $\eta|_{H^0(X,\mathcal{L})_{w_1}}$ is an isomorphism.

Proof. Since $C^+(x)$ is isomorphic to the affine space, its complement $D_x := X \setminus C^+(x)$ is a divisor (see [Goo69, Proposition 1]). Consequently the Cartier class group $\mathrm{Cl}(X)$ is freely generated by the irreducible components of D_x. By our assumption that X has Picard number one, D_x is the prime generator of $\mathrm{Cl}(X)$.

We can therefore write $\mathcal{L} \cong \mathcal{O}_X(r_0D_x)$ for some positive integer r_0. For any nonzero section s with $D_+(s) = C^+(x)$, write $Z(s) = rD_x$ for some positive integer r. Then $rD_x \sim r_0D_x$ implies $r = r_0$, and consequently the subspace
$$\{s \in H^0(X,\mathcal{L}) \mid D_+(s) = C^+(x)\}$$
is one-dimensional. This subspace is $\mathbb{C}^*$-invariant; we denote its weight by w'.

Fix a nonzero section s_0 in this subspace and define η as above. The map η is injective because $C^+(x)$ is open in X. For any $s \in H^0(X,\mathcal{L})$, $z \in \mathbb{C}^*$, and $u \in C^+(x)$, we compute:
$$\begin{aligned}(z\cdot s)(u) &= z\cdot s(z^{-1}\cdot u)\\ &= z\cdot\big(\eta(s)(z^{-1}\cdot u)s_0(z^{-1}\cdot u)\big)\\ &= \eta(s)(z^{-1}\cdot u)(z\cdot s_0)(u)\\ &= z^{w'}(z\cdot\eta(s))(u)s_0(u),\end{aligned}$$
which shows that $\eta(z\cdot s) = z^{w'}(z\cdot\eta(s))$.

For $s \in H^0(X,\mathcal{L})_{w_k}$, viewing $\eta(s)$ as a regular function on T_xX via $C^+(x) \cong T_xX$, we have:
$$\begin{aligned}\eta(z\cdot s)(u) &= z^{w_k}\eta(s)(u),\\ z\cdot\eta(s)(u) &= \eta(s)(z^{-1}\cdot u) = \eta(s)(z^{-1}u).\end{aligned}$$
Combining this with $\eta(z\cdot s) = z^{w'}(z\cdot\eta(s))$, we conclude that $\eta(s)$ is a homogeneous polynomial on T_xX of degree $w' - w_k \geq 0$. This forces $w' = w_0 = 0$ and $\deg(\eta(s)) = -w_k$.

Therefore, η is $\mathbb{C}^*$-equivariant, and for any nonzero $s \in H^0(X,\mathcal{L})_0$, $\eta(s)$ is constant. This proves claims (1) and (2).

For any section $s \in H^0(X,\mathcal{L})$, we express $\eta(s) = \sum_{k\geq 0}\eta_k(s)$ as a sum of homogeneous functions on $C^+(x) \cong T_xX$. The k-th jet of s at x corresponds to the class $[\eta(s)]$ in $\mathcal{O}_x/\mathfrak{m}_x^{k+1}$, which under our identification is represented by $\sum_{i=0}^{k}\eta_i(s)$.

Since η maps $H^0(X,\mathcal{L})_{w_k}$ to homogeneous polynomials of degree $-w_k$ on T_xX, and by our choice of isomorphism $C^+(x) \cong T_xX$ together with the definition of fundamental forms, we obtain:

$$\eta(H^0(X,\mathcal{L})_{w_k}) = \mathbb{F}_x^{-w_k} \quad \text{and} \quad \mathrm{Im}(\eta) = \mathbb{F}_x.$$

Finally, since $\mathcal{L}$ is very ample, [Har77, Proposition II.7.2] implies that $\mathrm{Im}(\eta)$ generates $\mathbb{C}[C^+(x)]$ as a $\mathbb{C}$-algebra. In particular, $\mathrm{Im}(\eta)$ contains the linear functions T_x^*X, which must coincide with the image of $H^0(X,\mathcal{L})_{w_1}$. We conclude that $w_1 = -1$ and $\eta|_{H^0(X,\mathcal{L})_{w_1}}$ is an isomorphism. □

By Proposition 3.5, the fundamental forms at x are given by the linear injections $\eta|_{H^0(X,\mathcal{L})_{w_k}}$ for each k. Dually, they are represented by the surjective linear maps

$$\Pi_k \colon \mathrm{Sym}^{-w_k}(T_xX) \to H^0(X,\mathcal{L})_{w_k}^\vee,$$

where $\Pi_0 \colon \mathbb{C} \to H^0(X,\mathcal{L})_0^\vee$ maps 1 to the unique $e_0 \in H^0(X,\mathcal{L})_0^\vee$ with $e_0(s_0) = 1$, and Π_1 is an isomorphism by Lemma 3.5(3). An immediate corollary of Theorem 3.3 and Proposition 3.5 is as follows:

Corollary 3.6. *In the setting of Definition 3.4 and Proposition 3.5, assume further that $x \in X$ is a general point. Then the following hold:*

1. $w_k = -k$ *for each* $0 \le k \le r$, $\eta(H^0(X,\mathcal{L})_{-k}) = \mathbb{F}_x^k$, *and* $\mathbb{F}_x = \oplus_{k=0}^r \mathbb{F}_x^k$.
2. *The projective embedding $f \colon X \to \mathbb{P}V$ (where $V = H^0(X,\mathcal{L})^\vee$) restricts on $C^+(x) \cong T_xX$ to* $f(u) = \left[\sum_{k=0}^{r} \Pi_k(u,\ldots,u)\right]$ *for any $u \in T_xX$, and $f(x) = f(0) = [e_0]$. for any $u \in T_xX$, with $f(x) = [e_0]$ at $u = 0$.*

In what follows, we keep the assumptions as in Corollary 3.6. The contraction homomorphism in Definition 3.2(2) induces a locally nilpotent action of T_xX on $\bigoplus_{k\ge 0} \mathrm{Sym}^k T_x^*X$. It follows from Theorem 3.3 that the system of funfamental forms $\mathbb{F}_x$ is a finite-dimensional T_xX-invariant subspace. Dually, this defines a nilpotent action of T_xX on $H^0(X,\mathcal{L})^\vee$, which we denote by Γ, via the maps Π_k. Explicitly, for each $0 \le k \le r$ and any $v \in T_xX$, we define

$$\Gamma_v|_{V_k} : V_k \to V_{k+1}$$

by the relations:

$$\Gamma_v \circ \Pi_k(v_1,\ldots,v_k) = \Pi_{k+1}(v, v_1,\ldots,v_k) \quad (k \ge 1), \tag{3.1}$$

with $\Gamma_v(e_0) = \Pi_1(v)$ and $\Gamma_v(V_r) = 0$. Consequently, the embedding $f|_{T_xX}$ defined in Corollary 3.6(2) can be expressed as:

$$f(u) = \left[\sum_{k=0}^{r} \Gamma_u^k(e_0)\right] \quad \text{for any } u \in T_xX. \tag{3.2}$$

Then the vector group action arises from a linear representation $\rho_x : T_xX \to \mathrm{GL}(V)$ given by:

$$\rho_x(u)(v) = \sum_{l=0}^{r}\sum_{k=0}^{l} \binom{l}{k} \Gamma_u^{l-k}(v_k), \tag{3.3}$$

for any $u \in T_xX$ and $v = \sum_{k=0}^{r} v_k \in V$ with $v_k \in V_k$. We summarize key properties:

Proposition 3.7. *Let $\mathcal{L}$ be very ample on X with a $\mathbb{C}^*$-action on $(X,\mathcal{L})$ that is Euler-type at a general fixed point x. In the notation as above, Then:*

1. *For any $k \ge 1$ and $u_1,\ldots,u_k \in T_xX$,*
$$\Pi_k(u_1,\ldots,u_k) = \Gamma_{u_1} \circ \cdots \circ \Gamma_{u_k}(e_0).$$
2. *The representation ρ_x induces a T_xX-action on $\mathbb{P}V$ preserving X and lifting the action on $C^+(x)$. Its differential $d\rho_x : T_xX \to \mathfrak{gl}(V)$ satisfies:*
$$d\rho_x|_{V_k} = (k+1)\Gamma|_{V_k},$$
with $d\rho_x(u) \cdot V_k \subseteq V_{k+1}$ for all $u \in T_xX$, and $d\rho_x(u)(e_0) = \Pi_1(u)$.

Proof. (1) follows immediately from the definition of Γ. For (2), the first claim is proved in [FH20, Proof of Theorem 3.7]. For the differential, take $u \in T_xX$ and $v = \sum_{k=0}^{r} v_k \in V$:

$$\begin{aligned} d\rho_x(u)(v) &= \frac{d}{dz}\bigg|_{z=0} \rho_x(zu)(v) \\ &= \frac{d}{dz}\bigg|_{z=0} \sum_{l=0}^{r}\sum_{k=0}^{l} \binom{l}{k} z^{l-k}\Gamma_u^{l-k}(v_k) \\ &= \sum_{l=1}^{r} l\Gamma_u(v_{l-1}). \end{aligned}$$

Thus $d\rho_x|_{V_k} = (k+1)\Gamma|_{V_k}$, yielding the weight conditions. □

Next, we apply it to the case when both the sink and the source are isolated. Assume that for a general pair of points x, y on X, there is a $\mathbb{C}^*$-action on X which is of Euler type at the source x, and its inverse action is of Euler type at y. Take the linearization of its inverse action on $\mathcal{L}$ such that the decomposition of the associated weight subspaces equals

$$H^0(X,\mathcal{L}) = \bigoplus_{k=0}^{r} H^0(X,\mathcal{L})'_{w'_k},$$

where $0 = w'_0 > \cdots > w'_r$. Applying Proposition 3.7, we have the following:

Corollary 3.8. *Let X, $\mathcal{L}$, x, and y be as above.*

(1) *We have $H^0(X,\mathcal{L})_{w_k} = H^0(X,\mathcal{L})'_{w'_{r-k}}$ and $w'_k = -r - w_{r-k} = -k$.*

(2) *We have*

$$V_r = H^0(X,\mathcal{L})'_{w'_0} = \{s \in H^0(X,\mathcal{L}) : D_+(s) = X^-(y)\},$$

and $\dim(H^0(X,\mathcal{L})'_{w'_0}) = 1$. Furthermore, $f(y) = [e_r]$, where e_r is a nonzero element in V_r.

(3) *There is a vector group action of T_yX on V, namely $\rho_y : T_yX \to \mathrm{GL}(V)$, such that the induced action on $\mathbb{P}V$ leaves X invariant and lifts the action of T_yX on $C^-(y)$. Moreover, for the differential map $d\rho_y$, we have $d\rho_y \cdot V_0 = 0$ and*

$$d\rho_y(w) \cdot V_{k+1} \subseteq V_k,$$

for any $w \in T_yX$ and for any $k \geq 0$.

Proof. (1) follows directly from the definition of w'_k.

(2) follows from (1), Proposition 3.5(1), and Proposition 3.7(2).

For (3), we dually denote $V'_k = (H^0(X,\mathcal{L})'_{w'_k})^\vee$ for each k. Then, by (1), we have $V'_k = V_{r-k}$. From Proposition 3.7(2), there is a vector group action of T_yX on V, such that the induced action on X extends the action of T_yX on $C^-(y)$. Moreover,

$$d\rho_y \cdot V'_k \subseteq V'_{k+1},$$

which is equivalent to

$$d\rho_y \cdot V_{k+1} \subseteq V_k$$

for any $k \geq 0$, and $d\rho_y \cdot V_0 = 0$. □

4. Prolongations of projective subvarieties

In this section, we study prolongation of projective subvarieties in two steps following [FH12] [HM05]. Firstly we study the prolongation of the VMRT of Euler-symmetric varieties at a general point (Proposition 4.5). Then we calculate the dimension of $\mathfrak{aut}(\hat{S})^{(1)}$ for certain $S \subset \mathbb{P}V$ with nonzero prolongation (Proposition 4.8), which was explicitly formulated in [FH12].

4.1. **Prolongations of the VMRT.** In this section, we study the prolongations of the VMRT for smooth Euler-symmetric varieties of Picard number one. The VMRT at a general point of such varieties relates to their fundamental forms as follows.

Definition 4.1. *Let X be a smooth projective variety of Picard number one. Assume that for a general point $x \in X$, there exists a $\mathbb{C}^*$-action on X that is of Euler type at x. Fix a very ample line bundle $\mathcal{L}$ on X with the normalized linearization and consider the projective embedding $f : X \to \mathbb{P}H^0(X, \mathcal{L})^\vee$ as in Section 3. Let $\mathbb{F}_x$ denote the fundamental forms of X at x, and take η as in Proposition 3.5. For each $k \geq 2$, define the base locus:*

$$\mathrm{Bs}(\mathbb{F}_x^k) = \{[w] \in \mathbb{P}T_xX \mid \phi(w, \dots, w) = 0 \text{ for all } \phi \in \mathbb{F}_x^k \subset \mathrm{Sym}^k T_x^* X\}.$$

By Proposition 3.5, this satisfies $\mathrm{Bs}(\mathbb{F}_x^k) = \{[w] \in \mathbb{P}T_xX \mid \eta(s)(w) = 0 \text{ for all } s \in H^0(X, \mathcal{L})_{-k}\}$.

Since $\mathbb{F}_x$ forms a symbol system, we have an increasing filtration:

$$\mathrm{Bs}(\mathbb{F}_x^2) \subset \mathrm{Bs}(\mathbb{F}_x^3) \subset \cdots \subset \mathrm{Bs}(\mathbb{F}_x^r).$$

The base locus of fundamental forms at x is defined as $\mathrm{Bs}(\mathbb{F}_x) = \mathrm{Bs}(\mathbb{F}_x^{l_0})$, where l_0 is the minimal integer such that $\mathrm{Bs}(\mathbb{F}_x^{l_0})$ is non-empty.

Proposition 4.2. *[FH20, Theorem 3.7, Propositions 4.4 and Prop 5.4(iii)]*
Let X be a smooth projective variety of Picard number one that admits a $\mathbb{C}^*$-action of Euler type at a general point $x \in X$. Fix a very ample line bundle $\mathcal{L}$ on X with associated fundamental forms $\mathbb{F}_x$ for the embedding $f : X \to \mathbb{P}H^0(X, \mathcal{L})^\vee$. Let $\mathcal{K}$ be the family of minimal rational curves containing $\mathbb{C}^*$-stable curves of minimal degree through x, and let $\mathcal{C} \subset \mathbb{P}T(X)$ be the corresponding VMRT structure (Definition 2.7). Then:

$$\mathcal{C}_x = \mathrm{Bs}(\mathbb{F}_x) \subset \mathbb{P}T_xX$$

is an irreducible, nonsingular, and non-degenerate projective subvariety.

Next, we study the prolongation of infinitesimal linear automorphisms of $\mathcal{C}_x \subseteq \mathbb{P}T_xX$ in the setting of Theorem 3.1. Assume that for a general pair of points (x, y) on X, there exists a $\mathbb{C}^*$-action that is of Euler type at x, and whose inverse action is of Euler type at y. We keep the notations as in Definition 3.4. Fix $\mathbb{C}^*$-isomorphisms $C^+(x) \cong T_xX$ and $C^-(y) \cong T_yY$ such that the tangent maps at x and y are the identity maps, respectively.

We fix a very ample line bundle $\mathcal{L}$ on X and set $V = H^0(X, \mathcal{L})^\vee$. Recall the weight decomposition:

$$V = \bigoplus_{k=0}^{r} V_k, \quad \text{where } V_k = H^0(X, \mathcal{L})^\vee_{-k},$$

and the surjective maps $\Pi_k : \mathrm{Sym}^k T_xX \to V_k$ for $0 \leq k \leq r$.

Let $f : X \to \mathbb{P}V$ be the projective embedding, and denote by ρ_x and ρ_y the linear actions of T_xX and T_yX on V, respectively (3.3). Define:

$$\mathfrak{g}_1 = \mathrm{Im}(d\rho_x) \subset \mathfrak{gl}(V), \quad \mathfrak{g}_{-1} = \mathrm{Im}(d\rho_y) \subset \mathfrak{gl}(V).$$

By construction, $\mathfrak{g}_1, \mathfrak{g}_{-1} \subset \mathfrak{aut}(\hat{X})$, so for any $\alpha \in \mathfrak{g}_1$ and $\beta \in \mathfrak{g}_{-1}$, the commutator $\gamma := [\alpha, \beta]$ also lies in $\mathfrak{aut}(\hat{X})$. Consider the one-parameter subgroup:

$$G_\gamma = \{\exp(z\gamma) \mid z \in \mathbb{C}\} \subset \mathrm{GL}(V).$$

This group action on $\mathbb{P}V$ preserves X. Moreover, we can explicitly compute the isotropy action of G_γ on T_xX as follows. Recall from Proposition 3.5(3) that $\Pi_1 : T_xX \to V_1$ is an isomorphism.

Proposition 4.3. *The points x and y are fixed by G_γ. Let $\Phi_\gamma \colon G_\gamma \to \mathrm{GL}(T_xX)$ denote the induced isotropy representation of G_γ on T_xX. Then, for any $w \in T_xX$, the following holds:*

$$\Pi_1\left((d\Phi_\gamma)(\gamma)(w)\right) = [\gamma, d\rho_x(w)] \cdot e_0 \in V_1. \tag{4.1}$$

Moreover, if $d\Phi_\gamma(\gamma) = 0$, then G_γ acts trivially on X.

Proof. By Theorem 3.1(2), the weight conditions

$$\alpha \cdot V_k \subset V_{k+1} \quad \text{and} \quad \beta \cdot V_{k+1} \subset V_k \quad \text{for each } 0 \leq k \leq r-1$$

imply that $\gamma \cdot V_k \subset V_k$ for all $0 \leq k \leq r$. Thus, G_γ preserves each V_k. Since $x = [e_0]$ and $y = [e_r]$, where V_0 and V_r are one-dimensional (by Theorem 3.1(1)), both x and y are fixed by G_γ. Furthermore, the G_γ-action commutes with the $\mathbb{C}^*$-action.

For any nonzero tangent vector $w \in T_xX$, identified with an element of $C^+(x)$, consider the holomorphic arc θ_w through x (see (3.2)):

$$\begin{aligned}\theta_w\colon \mathbb{C} &\to X \subset \mathbb{P}V,\\ z &\mapsto f(zw) = \left[\sum_{k=0}^{r} z^k \Gamma_w^k(e_0)\right].\end{aligned}$$

Let C_w denote the closure of its image. Then C_w is a nontrivial $\mathbb{C}^*$-orbit closure with source x. Since G_γ commutes with $\mathbb{C}^*$, for any $g \in G_\gamma$, the curve $g \cdot C_w$ is also a $\mathbb{C}^*$-orbit closure through $g \cdot x = x$. Hence, $g \cdot f(w) \in C^+(x)$ for all w, meaning $g \cdot C^+(x) \subset C^+(x)$.

Via the isomorphism $C^+(x) \cong T_xX$, this induces a linear action of G_γ on T_xX as follows. Writing $\gamma \cdot e_0 = ce_0$ for some $c \in \mathbb{C}$, we compute for $g_t = \exp(t\gamma)$:

$$\begin{aligned}f(g_t \cdot w) &= \left[\sum_{k=0}^{r} \Gamma^k_{g_t\cdot w}(e_0)\right] = \left[e_0 + \Gamma^1_{g_t\cdot w}(e_0) + \sum_{k=2}^{r} \Gamma^k_{g_t\cdot w}(e_0)\right],\\ g_t \cdot f(w) = g_t \cdot &\left[\sum_{k=0}^{r} \Gamma^k_w(e_0)\right] = \left[e_0 + e^{-tc} g_t \cdot \Gamma^1_w(e_0) + \sum_{k=2}^{r} e^{-tc} g_t \cdot \Gamma^k_w(e_0)\right].\end{aligned}$$

From the definition we have $f(g_t \cdot w) = g_t \cdot f(w)$, we deduce:

$$\Pi_1(g_t \cdot w) = \Gamma^1_{g_t\cdot w}(e_0) = e^{-tc} g_t \cdot \Pi_1(w). \tag{4.2}$$

Since the differential of the isomorphism $T_xX \cong C^+(x)$ at $0 \in T_xX$ is the identity map, this action coincides with the isotropy representation Φ_γ. Therefore:

$$\begin{aligned}\Pi_1\left((d\Phi_\gamma)(\gamma)(w)\right) &= \Pi_1\Big(\frac{d}{dt}\Big|_{t=0} (g_t \cdot w)\Big) = \frac{d}{dt}\Big|_{t=0} \Pi_1(g_t \cdot w)\\ &= \frac{d}{dt}\Big|_{t=0} \left(e^{-tc} g_t \cdot \Pi_1(w)\right) = -c\Pi_1(w) + \gamma \cdot \Pi_1(w) = [\gamma, d\phi_x(w)] \cdot e_0,\end{aligned}$$

where the first equality holds as Π_1 is linear, and the last equality follows from $\Pi_1(w) = \Gamma_w(e_0) = d\phi_x(w) \cdot e_0$.

For the final statement, observe that the action of G_γ on $T_xX \cong C^+(x)$ is given by the exponential of its Lie algebra. From (4.2), it corresponds to the action of

$$e^{-tc} g_t = e^{t(\gamma|_{V_1} - c\cdot \mathrm{Id}|_{V_1})}$$

on V_1, where the latter is the exponential of $t(d\Phi_\gamma(\gamma))$ via Π_1. Thus, if $d\Phi_\gamma(\gamma) = 0$, then $\Phi_\gamma(G_\gamma) = \mathrm{Id}$. Thus G_γ acts trivially on $C^+(x) \cong T_xX$, consequently on X since $C^+(x)$ is open and dense. ☐

As G_γ fixes x, it acts on the family of minimal rational curves through x. Thus the image of Φ_γ is contained in $Aut^0(\hat{\mathcal{C}_x})$ and $d\Phi_\gamma(\gamma) \in \mathfrak{aut}(\hat{\mathcal{C}_x})$. Then under the identification Π_1, we can rewrite Lemma 4.3 as follows, which is a generalization of the map (2.1).

Corollary 4.4. *Consider the linear map $\lambda : \mathfrak{g}_{-1} \to Sym^2(V_1^*) \otimes V_1$ given by:*

$$\begin{aligned}\lambda(d\rho_y(\beta)) : \quad V_1 \times V_1 &\longrightarrow V_1\\ (\Pi_1(\alpha), \Pi_1(\xi)) &\longrightarrow [[d\rho_y(\beta), d\rho_x(\alpha)], d\rho_x(\xi)].e_0,\end{aligned} \tag{4.3}$$

for any $\beta \in T_yX$ and for any $\alpha, \xi \in T_xX$. Then $Im(\lambda) \subset \mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)}$, under the identiication Π_1.

Proof. $\lambda(d\rho_y(\beta))$ is symmetric because $\mathfrak{g}_1 = \mathrm{Im}(d\rho_x)$ is abelian. For any fixed $\alpha \in T_xX$, $\beta \in T_yX$, denote $\gamma = [d\rho_y(\beta), d\rho_x(\alpha)]$ as above. We check that $\lambda(d\rho_y(\beta))(\Pi_1(\alpha)) \in \mathfrak{aut}(\hat{\mathcal{C}_x})$ under Π_1:

$$\lambda(d\rho_y(\beta))(\Pi_1(\alpha))(\Pi_1(\xi)) = [\gamma, d\rho_x(\xi)].e_0 = \Pi_1\left((d\Phi_\gamma)(\gamma)(\xi)\right),$$

where the first equality is by definition, and the second equality is from (4.1) in Proposition 4.3.

☐

The technical heart of our article is the following observation.

Proposition 4.5. *λ induces a bijection from $\mathfrak{g}_{-1}$ onto $\mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)}$.*

Proof. Assume that $\mathcal{C}_x \subset \mathbb{P}T_xX$ is linearly normal. Then by Theorem 2.11(2), there is a natural inclusion:

$$\mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)} \hookrightarrow T_x^*X.$$

From Proposition 4.8 below, any irreducible, nonsingular and non-degenerate projective subvariety $S \subset \mathbb{P}W$ satisfying $\dim(\mathfrak{aut}(S)^{(1)}) \geq \dim(V)$ must be linearly normal. Thus it suffices to show that λ is injective.

Assume $\lambda(d\rho_y(\beta)) = 0$ for some nonzero $\beta \in T_yX$. For any $\alpha \in T_xX$, define $\gamma_\alpha = [d\rho_y(\beta), d\rho_x(\alpha)]$. We show $\gamma_\alpha = l(\alpha)\mathrm{Id}_V$ for some $l \in T_x^*X$ by proving G_{γ_α} acts by scalars. By Proposition 4.3 it implies that $d\Phi_{\gamma_\alpha}(\gamma_\alpha) = 0$, so G_{γ_α} acts trivially on X. Thus any $v \in \hat{X}$ is a G_{γ_α}-eigenvector. For any $g \in G_{\gamma_\alpha}$, we denote its eigenspaces in V by $\{V_{g(c)} : c \in J_g\}$, where J_g is a finite set of indices. Then we have $X \subset \cup_{c \in J_g} \mathbb{P}V_{g(c)}$. As X is non-degenerate and irreducible, some $V_{g(c)} = V$, so g acts by scalars.

Let $U_\beta = \{\exp(d\rho_y(t\beta)) : t \in \mathbb{C}\} \subset Im(\rho_y)$ and $V_l = \{\exp(d\rho_x(\alpha)) : l(\alpha) = 0\} \subset Im(\rho_x)$. By the definition of l it implies that U_β commutes with V_l. By Corollary 3.8, $x = [e_0]$ is fixed by U_β since $d\rho_y(\beta) \cdot e_0 = 0$. The commuting actions imply $V_l \cdot x$ is also U_β-fixed. But U_β, as a subgroup of T_yY, acts freely on $C^-(y) \cong T_yX$, forcing $V_l \cdot x \subset C^+(x) \setminus C^-(y)$.

We derive contradictions:

- If $l = 0$, then $V_l = Im(\rho_x)$ and $V_l \cdot x = C^+(x)$. But $C^+(x) \cap C^-(y) \neq \emptyset$ as both are open dense.
- If $l \neq 0$, then $V_l \cdot x$ is a hyperplane in $C^+(x) \cong T_xX$. By Proposition 3.5, $D_y := X \backslash C^-(y)$ is an irreducible divisor, so $C^+(x) \backslash C^-(y)$ is an irreducible divisor of $C^+(x)$ and equals $V_l \cdot x$.

  Corollary 3.8 gives $C^+(x) \backslash C^-(y) = \{w \in T_xX : \eta(s_r)(w) = 0\}$ where $s_r \in H^0(X, \mathcal{L})_{w_r}$. Thus $Bs(\mathbb{F}_x^r) = \{[w] \in \mathbb{P}T_xX : \eta(s_r)(w) = 0\}$ is a hyperplane. But then $\mathcal{C}_x = Bs(\mathbb{F}_x) \subset Bs(\mathbb{F}_x^r)$ is linearly degenerate in $\mathbb{P}T_xX$, contradicting Proposition 4.2.

□

4.2. **Projective subvarieties with nonzero prolongations.** Let us recall the classification result of projective subvarieties with nonzero prolongations by Fu and Hwang as follows.

Theorem 4.6. *[FH12, Main Theorem] and [FH18, Theorem 7.13] Let $S \subset \mathbb{P}V$ be an irreducible nonsingular non-degenerate variety such that $\mathfrak{a}ut(\hat{S})^1 \neq 0$. Then $S \subset \mathbb{P}V$ is projectively equivalent to one of the followings:*

(1) The VMRT of an IHSS of rank $\geqslant 2$.

(2) The VMRT of a symplectic Grassmannian.

(3) A nonsingular linear section of $Gr(2,5) \subset \mathbb{P}^9$ of codimension $\leqslant 2$.

(4) A nonsingular $\mathbb{P}^4$-general linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ of codimension $\leqslant 3$.

(5) Biregular projections of (1) and (2) with nonzero prolongations, which are completely described in Section 4 of [FH12] .

Remark 4.7. As noted in [FH18, Proposition 2.11], all nonsingular sections of $Gr(2,5) \subset \mathbb{P}^9$ with codimension $s \leqslant 3$ are projectively equivalent.

The main result of this subsection is the following result based on Theorem 4.6.

Proposition 4.8. *Let $S \subset \mathbb{P}V$ be one of the projective subvarieties in Theorem 4.6 (2)(3)(4)(5), then:*

$$dim(\mathfrak{aut}(\hat{S})^{(1)}) < dim(V). \tag{4.4}$$

We will prove this proposition case by case based on Theorem 4.6.

4.2.1. *Case (2) and (5).* In these cases, the prolongation of $\mathfrak{a}ut(\hat{S})$ was explicitly formulated in [FH12]. First we consider Case (2) and the case of biregular projections of (2):

Lemma 4.9. *Let W and Q be vector spaces of dimensions $k \geqslant 2$ and m respectively. Set $L = Sym^2(Q) \subset V = Sym^2(W \oplus Q)$ and $U = V/L$. For $\phi \in Sym^2(W \oplus Q)$ denote by $\phi^\# \in Hom(W^\vee \oplus Q^\vee, W \oplus Q)$ the corresponding homomorphism via the natural inclusion $Sym^2(W \oplus Q) \subset Hom((W^\vee \oplus Q^\vee, W \oplus Q)$, and denote by $\overline{\phi}$ its image in U. For $L_2 \subset U$, let $Im(L_2)$ be the linear space of $\{Im(\phi^\#) : \overline{\phi} \in L_2\}$. Define $Im_W(L_2) = P_Q(Im(L_2)) \subset W$, where $P_Q : W \oplus Q \to W$ is the projection to the first factor, then:*

(i) Denote by $p_L : \mathbb{P}V \dashrightarrow \mathbb{P}(V/L)$ the projection from $\mathbb{P}L$. Let $v_2 : \mathbb{P}(W \oplus Q) \to \mathbb{P}(Sym^2(W \oplus Q))$ be the second Veronese embedding, Z the proper image of $Im(v_2)$. Then $Z \subset \mathbb{P}V/L = \mathbb{P}U$ is isomorphic to the VMRT of the symplectic Grassmannian $Gr_w(k, \Sigma)$ at a general point and $\mathfrak{aut}(\hat{Z})^{(1)} \cong Sym^2(W^\vee)$.

(ii) If $Z \cap \mathbb{P}L_2 = \emptyset$, then $\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)} \cong Sym^2(W/Im_W(L_2))^\vee$.

(iii) $dim(\mathfrak{aut}(\hat{Z})^{(1)}) < dim(V/L)$. *Let* $L_2 \subset U$ *be as in (ii), if* $\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)} \neq 0$, *then:*

$$dim(\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)}) < dim(U/L_2).$$

Proof. (i) and (ii) are from [FH12, Proposition 4.18].

Under the identification: $V = Sym^2(W \oplus Q) \subset Hom(W^\vee \oplus Q^\vee, W \oplus Q)$, we write $U = V/L = Sym^2(W) \oplus Hom(Q^\vee, W)$, where we identify $Sym^2(W)$ inside $Hom(W^\vee, W)$. Thus $dim(V/L) > dim(\mathfrak{aut}(\widehat{Z})^{(1)})$ as $Hom(Q^\vee, W) \neq 0$. Take a basis of $Im_W(L_2)$ to be $e_1, ..., e_t$ and extend it to a basis of W: $e_1, ..., e_t, e_{t+1}, ..., e_k$. Then by the definition of $Im_W(L_2)$, we have:

$$\begin{aligned} L_2 \subset &\{(\phi, \eta) \in Sym^2(W) \oplus Hom(Q^\vee, W) \mid Im(\phi^\#) \subset Im_W(L_2) \text{ and } Im(\eta) \subset Im_W(L_2)\} \\ &\cong Sym^2(Im_W(L_2)) \oplus Hom(Q^\vee, Im_W(L_2)), \end{aligned}$$

whence $dim(L_2) \leqslant \dfrac{t(t+1)}{2} + mt$. Then:

$$\begin{aligned} dim(U/L_2) - dim(\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)}) &\geqslant \frac{k(k+1)}{2} + km - \frac{t(t+1)}{2} - tm - \frac{(k-t)(k-t+1)}{2} \\ &= (m+t)(k-t) > 0, \end{aligned}$$

where $k - t > 0$ as $\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)} \neq 0$. ☐

By [FH12, Main Theorem (C)], the other cases of (5) are biregular projections of the VMRT of $Gr(a, a+b), \mathbb{S}_n$ and $Lag(n, 2n)$ respectively. We prove these cases by the following three lemmas.

Lemma 4.10. *Let A and B be vector spaces with $a = dim(A) \geqslant b = dim(B) \geqslant 3$. Let $V = Hom(A, B)$. For a subspace $L \subset V$, set $Im(L) = \{Im(\phi) \subset B : \phi \in L\}$. $Ker(L) = \bigcap_{\phi \in L} Ker(\phi)$. Then:*

(i) $S = \{[\phi] \in \mathbb{P}V : rank(\phi) \leqslant 1\} \subset \mathbb{P}V$ is projectively isomorphic to the VMRT of $Gr(a, a+b)$.

(iii) Let $L \subset V$ such that $L \cap Sec(S) = \emptyset$, then $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong Hom(B/Im(L), Ker(L))$

(iii) Let $L \subset V$ be as in (ii), if $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \neq 0$ then $dim(\mathfrak{aut}(\widehat{p_L(S)})^{(1)}) < dim(V/L)$.

Proof. (i) and (ii) are from [FH12, Proposition 4.10]. For (iii), denote $s = dim(Ker(L))$ and $t = dim(Im(L))$, by the definition we have

$$L \subset \{\phi \in Hom(A, B) : \phi|_{Ker(L)} = 0, Im(\phi) \subset Im(L)\} = Hom(A/Ker(L), Im(L)),$$

implying $dim(L) \leqslant (a-s)t$. Thus

$$dim(V/L) - dim(\mathfrak{aut}(\widehat{p_L(S)})^{(1)}) = ab - dim(L) - (b-t)s \geqslant ab - (a-s)t - (b-t)s = (a-t)(b-t) + st > 0,$$

where $s < a$ and $t < b$ as $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \neq 0$. ☐

Lemma 4.11. *Let W be a vector space of dimension $n \geqslant 6$. $V = \wedge^2 W$. For each $\phi \in \wedge^2 W$, denote by $\phi^\# \in Hom(W^\vee, W)$ via the inclusion $\wedge^2 W \subset W \otimes W = Hom(W^\vee, W)$. For a subspace $L \subset V$, define $Im(L) \subset W$ as the linear span of $\{Im(\phi^\#) \subset W, \phi \in L\}$. Then:*

(i) $S = \{[\phi] \in V : rk(\phi) \leqslant 2\} \subset \mathbb{P}V$ is isomorphic to the VMRT of $\mathbb{S}_n$.

(ii) If $L \subset V$ such that $\mathbb{P}L \cap Sec(S) = \emptyset$, then $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong \wedge^2(W/Im(L))^\vee$.

(iii) Let $L \subset V$ be as in (ii), if $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \neq 0$, then $dim(\widehat{p_L(S)})^{(1)}) < dim(V/L)$.

Proof. (i) and (ii) are from [FH12, Proposition 4.11]. For (iii) take a basis of $Im(L)$ to be $e_1, ..., e_t$ and extend it to a basis of W: $e_1, ..., e_t, e_{t+1}, ..., e_n$. Denote the dual basis of $W^\vee$ to be $f_1, ..., f_n$ such that $f_i(e_j) = \delta_{i,j}$ for any $1 \leqslant i, j \leqslant n$. Identify $Hom(W^\vee, W)$ with $M_{n \times n}(\mathbb{C})$ through:

$$\begin{aligned} Hom(W^\vee, W) &\longrightarrow M_{n\times n}(\mathbb{C}) \\ \mathcal{A} &\longrightarrow A = (a_{ij} : 1 \leqslant i, j \leqslant n) \end{aligned} \tag{4.5}$$

such that $\mathcal{A}(f_i) = \sum_{j=1}^{n} a_{ij} e_j$. Then V corresponds to all skew-symmetric matrices. Now

$$L \subset \{\mathcal{A} \in V : Im(\mathcal{A}) \subset Im(L)\} = \{\mathcal{A} \in V : a_{ij} = 0 \text{ if } \; i \geqslant r+1 \text{ or } j \geqslant r+1\},$$

thus $dim(L) \leqslant \dfrac{t(t-1)}{2}$. Then:

$$\begin{aligned} dim(V/L) - dim(\mathfrak{aut}(\widehat{p_L(S)})^{(1)}) &= \frac{n(n-1)}{2} - dim(L) - \frac{(n-t)(n-t-1)}{2} \\ &\geqslant \frac{n(n-1)}{2} - \frac{t(t-1)}{2} - \frac{(n-t)(n-t-1)}{2} = t(n-t) > 0, \end{aligned}$$

where $t > 0$ as $L \neq 0$ and $t < n$ as $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \neq 0$. □

Lemma 4.12. *Let W be a vector space of dimension $n \geqslant 3$. $V = Sym^2W$. For each $\phi \in Sym^2W$, denot by $\phi^{\#} \in Hom(W^\vee, W)$ via the inclusion $Sym^2W \subset W \otimes W = Hom(W^\vee, W)$. For a subspace $L \subset V$, define $Im(L) \subset W$ as the linear span of $\{Im(\phi^{\#}) \subset W, \phi \in L\}$. Then:*
(i) $S = \{[\phi] \in V : rk(\phi) \leqslant 1\} \subset \mathbb{P}V$ is isomorphic to the VMRT of $Lag(n, 2n)$.
(ii) If $L \subset V$ such that $\mathbb{P}L \cap Sec(S) = \emptyset$, then $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong Sym^2(W/Im(L))^\vee$.
(iii) Let $L \subset V$ be as in (ii), if $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \neq 0$, then $dim(\widehat{p_L(S)})^{(1)}) < dim(V/L)$.

Proof. (i) and (ii) are from [FH12, Proposition 4.12] . For (iii), as in Lemma 3.11 we take a basis of $Im(L)$ to be $e_1, ..., e_r$ and extend it to a basis of W to be $e_1, ..., e_r, e_{r+1}, ..., e_n$. Denote the dual basis to be $f_1, ..., f_n$. Keep the identification (3.5) then V corresponds to all symmetric matrices and we have

$$L \subset \{\mathcal{A} \in V : Im(\mathcal{A}) \subset Im(L)\} = \{\mathcal{A} \in V : a_{ij} = 0 \text{ if } \ i \geqslant r+1 \text{ or } j \geqslant r+1\},$$

implying $dim(L) \leqslant \dfrac{r(r+1)}{2}$ and thus

$$\begin{aligned} dim(V/L) - dim(\mathfrak{aut}(\widehat{p_L(S)})^{(1)}) &= \frac{n(n+1)}{2} - dim(L) - \frac{(n-r)(n-r+1)}{2} \\ &\geqslant \frac{n(n+1)}{2} - \frac{r(r+1)}{2} - \frac{(n-r)(n-r+1)}{2} = r(n-r) > 0, \end{aligned}$$

where $r > 0$ as $L \neq 0$ and $r < n$ as $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \neq 0$. □

4.2.2. *Case (3).* Let $X = Gr(2,5) \subset \mathbb{P}^9$. For each $k = 1, 2$, denote by $X_k \subset \mathbb{P}^{9-k}$ the nonsingular linear section of codimension k. Then the case when $S = X_1$ follows from [FH12, Section 3.4] and the case of X_2 follows from [BFM20, Lemma 4.6].

4.2.3. *Case (4).* Let $X = \mathbb{S}_5 \subset \mathbb{P}^{15}$. For each $k = 1, 2, 3$, denote by $X_k \subset \mathbb{P}^{15-k}$ the nonsingular $\mathbb{P}^4$-general linear section of codimension k as described in [FH18, Proposition 2.12].
(i) The case when $S = X_1$ follows from [FH12, Section 3.3].
(ii) By [FH18, Proposition 7.6] X_3 is quadratically symmetric. The VMRT of X_3 at a general point is a nonsingular linear section of $Gr(2,5) \subset \mathbb{P}^9$ of codimension 3, which has zero prolongations by Theorem 4.6. Then by the proof of [FH12, Theorem 6.15] we conclude that $\mathfrak{aut}(\hat{X}_3) \cong \mathbb{C}$.
(iii) To prove the case when $S = X_2$ we recall the following characterization of X_2 proved by Kuznetsov.

Theorem 4.13. *[Kuz18, Proposition 6.1 and Lemma 6.7] Let $X_K \subset X$ be a nonsingular linear section of X of codimension 2, then the followings are equivalent:*
(a) X_K is projectively equivalent to X_2;
(b) The Hilbert space $F_4(X_K)$ of linear 4-spaces on X_K is non-empty;
(c) There exists a line L in X_K such that

$$\mathcal{N}_{L/X_K} \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(1)^{\oplus 6} \tag{4.6}$$

Moreover, such line is unique and is equal to the intersection of all linear 4-spaces on X_K.

Now assume, for the sake of contradiction, that $\dim(\mathfrak{aut}(\hat{X}_2)^{(1)}) = \dim(V)$. Take L as the line defined in Theorem 4.13. For any point $x = [\hat{x}] \in L$, take a linear function $l \in V^\vee$ such that $l(\hat{x}) \neq 0$. Then, by Theorem 2.11(2), there exists a $\mathbb{C}^*$-action on X_2 which is of Euler type at x. We shall deduce a contradiction from this $\mathbb{C}^*$-action using the following lemmas.

Lemma 4.14. *There are exactly three different weight subspaces of the $\mathbb{C}^*$-action on V.*

Proof. From Theorem 2.11(2), the linear action of $\mathbb{C}^*$ on V has at most three different weight subspaces. On the other hand, if it admits only two different weights, then from Proposition 3.5(3), one easily sees that X_2 is isomorphic to a projective space, which leads to a contradiction. □

Under the setting of Definition 3.4 we have $r = 2$. Denote by $W = V_1$, $U = V_2$ and $f : X_2 \to \mathbb{P}V$ the projective embedding. Then we have: $dim(V) = 14$, $dim(W) = dim(T_xX_2) = 8$ and $dim(U) = dim(V) - 1 - dim(W) = 5$. We now check that the $\mathbb{C}^*$-action on X_2 satisfies the following two properties.

Lemma 4.15. *The $\mathbb{C}^*$-action on X_2 has exactly three irreducible components of $X_2^{\mathbb{C}^*}$: the isolated source $\{x\}$, the unique component Y_1 contained in $\mathbb{P}W$, and the unique component Y_2 contained in $\mathbb{P}U$.*

Proof. First, note that for any fixed component $Y \in \mathcal{Y}$, $Y \subset \mathbb{P}V_k$ for some k. If $Y \subset \mathbb{P}W$, then a $\mathbb{C}^*$-orbit whose sink lies in Y has its source equal to x. Thus, by [ORSW22, Lemma 2.8] and [RW20, Lemma 3.5], such Y is unique, and $C^-(Y)$ is a line bundle over Y. If $Y \subset \mathbb{P}U$, we see that $v^+(Y) = 0$, whence Y is the unique sink of the $\mathbb{C}^*$-action. □

Lemma 4.16. *We have* $\dim(Y_1) > 0$.

Proof. Assume that $Y_1 = \{y\}$ is a single point. If $\dim(Y_2) = 0$, then by [RW20, Lemma 3.5] we have $v^+(Y_1) = v^-(Y_1) = 1$, and thus $\dim(X_2) = \dim(T_yS_2) = v^+(Y_1) + v^-(Y_1) = 2 < 8$, which is a contradiction. If $\dim(Y_2) > 0$, then we have $D_x = X_2 \backslash C^+(x) = C^+(y) \cup Y_2$. This implies that Y_2, as a divisor of D_x, is of dimension 6, contradicting the fact that $Y_2 \subset \mathbb{P}U \cong \mathbb{P}^4$. □

Now, since L is the intersection of all linear 4-spaces in X_2, it is $\mathbb{C}^*$-invariant. Moreover, as $x \in L$ and the action is of Euler type at x, we conclude that L is a non-trivial $\mathbb{C}^*$-orbit closure with source $x = [e_0]$. Denote the orbit by $\mathbb{C}^* \cdot f(w) = \{[e_0 + z\Pi_1(w) + z^{-w_2}\Pi_2(w,w)] : z \in \mathbb{C}^*\}$ for some nonzero $w \in T_xX_2$, and denote by y the sink of the orbit. Then we must have $\Pi_2(w,w) = 0$, and $y \in Y_1$. Otherwise, the sink of the orbit would be $[\Pi_2(w,w)] \in \mathbb{P}U$. In that case, the line L would be contained in $\mathbb{P}(\mathbb{C}e_0 \oplus U)$, contradicting the fact that $\Pi_1(w) \neq 0$ as Π_1 is injective by Proposition 3.5(3).

Now take any point $y' \in Y_1$, and denote by $L_{y'}$ the unique non-trivial $\mathbb{C}^*$-orbit closure with sink y' and source x. Then $L_{y'}$ is exactly the line connecting x and y'. By [ORSW22, Lemma 2.16] and the proof of [ORSW22, Proposition 2.17], the splitting type of $T_{X_2}|_{L_{y'}}$ is determined by the weights of the isotropy action of $\mathbb{C}^*$ on $T_{y'}X_2$. As Y_1 is irreducible, the weights of $\mathbb{C}^*$ on $T_{y'}X_2$ remain invariant as y' varies in Y_1. This implies that the splitting type of $T_{X_2}|_{L_{y'}}$ also remains invariant, contradicting the uniqueness of L as $\dim(Y_1) > 0$.

Thus, we conclude that $\dim(\mathfrak{aut}(\hat{X}_2)^{(1)}) < \dim(V)$. This completes the proof of Proposition 4.8.

5. Proof of the main result

In this section we will prove Theorem 1.2. First Theorem 2.8 enables us to characterize an Euler-symmetric variety by its VMRT. We present a $\mathbb{C}^*$-equivariant version for our convenience.

Corollary 5.1. *Let X, X' be two Fano manifolds of Picard number 1. Assume that for a general point x on X and for a general point x' on X', there are $\mathbb{C}^*$-actions on X and X' such that the actions are of Euler type at x and x', respectively. Denote by $\mathcal{K}, \mathcal{K}'$ the families of minimal rational curves on X and X', which contain the $\mathbb{C}^*$-stable curves of minimal degree through x and x', respectively. Let $\mathcal{C} \subset \mathbb{P}T(X)$ and $\mathcal{C}' \subset \mathbb{P}T(X')$ be the associated VMRT structures on X and X'. If $\mathcal{C}_x$ is projectively isomorphic to ${\mathcal{C}'}_{x'}$, then there exists a $\mathbb{C}^*$-equivariant isomorphism $\Psi : X \to X'$ that maps x to x'.*

Proof. Assume that the projective isomorphism between $\mathcal{C}_x \subset \mathbb{P}T_xX$ and ${\mathcal{C}'}_{x'} \subset \mathbb{P}T_{x'}X'$ is induced by a linear isomorphism $\psi : T_xX \to T_{x'}X'$. Then, the identifications $C^+(x) \cong T_xX$ and $C^+(x') \cong T_{x'}X'$ induce an isomorphism $\overline{\psi} : C^+(x) \to C^+(x')$. By Proposition 4.2, their VMRTs at general points are both irreducible and nonsingular.

To extend $\overline{\psi}$, it suffices to check that the differential map of $\overline{\psi}$ preserves the VMRT at a general point. Since x and x' are both general points, by Proposition 3.7(2), the action of T_xX on $C^+(x)$ and the action of $T_{x'}X'$ on $C^+(x')$ can be extended to actions on X and X', respectively. This implies that the VMRT structures over $C^+(x)$ and $C^+(x')$ are locally flat. Specifically, the isomorphism $C^+(x) \cong T_xX$ yields an identification $\mathbb{P}T(X)|_{C^+(x)} \cong C^+(x) \times \mathbb{P}T_xX$, under which the VMRT structure corresponds to $\mathcal{C}|_{C^+(x)} \cong C^+(x) \times \mathcal{C}_x$ (and similarly for X').

Thus, by the definition of $\overline{\psi}$, for any point $y \in C^+(x)$, the differential map $d\overline{\psi}_y$ must map $\mathcal{C}_y$ isomorphically onto ${\mathcal{C}'}_{\overline{\psi}(y)}$. This guarantees the existence of an extension $\Psi : X \to X'$.

Finally, by our assumption that $\overline{\psi}$ is $\mathbb{C}^*$-equivariant, it follows that Ψ is also $\mathbb{C}^*$-equivariant, as $C^+(x)$ is open and dense in X. □

We are ready to prove our main result.

Proof of Theorem 1.2. If $X = G/P_\alpha$ is an IHSS of tube type as in Remark 2.13(3), then X is the equivariant compactification of the vector groups $R_u(P_\alpha^-)$ and $R_u(P_\alpha)$ respectively, where $x = [P_\alpha]$ is fixed by $R_u(P_\alpha)$ and $y = w_0 \cdot x$ is fixed by $R_u(P_\alpha^-)$. Consider the morphism

$$R_u(P_\alpha) \times R_u(P_\alpha^-) \to X \times X, \quad (u, v) \mapsto (uv \cdot x, u \cdot y).$$

Its image is dense and constructible; hence, it contains a dense open subset. Moreover, for any element $(uv \cdot x, u \cdot y)$ in the image, we can define a $\mathbb{C}^*$-action on X by:

$$\mathbb{C}^* \times X \longrightarrow X, \quad (t, x) \longmapsto uv\, t\, (uv)^{-1} \cdot x,$$

such that the $\mathbb{C}^*$-action is of Euler type at the source $\tilde{x} = uv \cdot x$ and its inverse action is of Euler type at $\tilde{y} = uv \cdot y = u \cdot y$.

Conversely, assume that for a general pair of points x, y on X, there is a $\mathbb{C}^*$-action which is of Euler type at x and whose inverse action is of Euler type at y. Let $\mathcal{K}$ be the family of minimal rational curves containing $\mathbb{C}^*$-stable curves of minimal degree through x, and let $\mathcal{C} \subset \mathbb{P}T(X)$ be the associated VMRT structure. By Theorem 1.3, for the projective embedding $\mathcal{C}_x \subset \mathbb{P}T_xX$, we have $\dim(\mathfrak{aut}(\hat{\mathcal{C}_x})^{(1)}) = \dim(T_xX)$.

Thus, by Theorem 4.6 and Proposition 4.8, the variety must be projectively isomorphic to the VMRT of an IHSS. Let us denote this IHSS by X'. Consider the $\mathbb{C}^*$-action defined in Remark 2.13(2), for which x' is an isolated source. The quadruple (X, x, X', x') then satisfies the hypotheses of Corollary 5.1, implying that X is $\mathbb{C}^*$-equivariantly isomorphic to X'. Moreover, since the $\mathbb{C}^*$-action on X has an isolated sink and an isolated source, the isomorphism forces X' to be an IHSS of tube type, as characterized in Remark 2.13(3). □

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References

[Arz11] I. V. Arzhantsev, *Flag varieties as equivariant compactifications of* $\mathbb{G}_a^n$, Proc. Amer. Math. Soc. **139** (2011), no. 3, 783–786.

[BFM20] C. Bai, B. Fu, and L. Manivel, *On Fano complete intersections in rational homogeneous varieties*, Math. Z. **295** (2020), no. 1-2, 289–308.

[BB73] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[Bri15] M. Brion, *On linearization of line bundles*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 1, 113–147.

[FH12] B. Fu and J.-M. Hwang, *Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity*, Invent. Math. **189** (2012), no. 2, 457–513.

[FH18] B. Fu and J.-M. Hwang, *Special birational transformations of type* (2, 1), J. Algebraic Geom. **27** (2018), no. 1, 55–89.

[FH20] B. Fu and J.-M. Hwang, *Euler-symmetric projective varieties*, Algebr. Geom. **7** (2020), no. 3, 377–389.

[Goo69] J. E. Goodman, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. (2) **89** (1969), 160–183.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York-Heidelberg, 1977.

[HM01] J.-M. Hwang and N. Mok, *Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1*, J. Math. Pures Appl. (9) **80** (2001), no. 6, 563–575.

[HM05] J.-M. Hwang and N. Mok, *Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation*, Invent. Math. **160** (2005), no. 3, 591–645.

[Ive72] B. Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.

[Kuz18] A. Kuznetsov, *On linear sections of the spinor tenfold. I*, Izv. Math. **82** (2018), no. 4, 694–751.

[LM03] J. M. Landsberg and L. Manivel, *On the projective geometry of rational homogeneous varieties*, Comment. Math. Helv. **78** (2003), no. 1, 65–100.

[ORSW22] G. Occhetta, E. A. Romano, L. E. Solá Conde, and J. A. Wiśniewski, *Small bandwidth* $\mathbb{C}^*$*-actions and birational geometry*, J. Algebraic Geom. **32** (2023), no. 1, 1–57.

[ORSW21] G. Occhetta, E. A. Romano, L. E. Solá Conde, and J. A. Wiśniewski, *Rational homogeneous spaces as geometric realizations of birational transformations*, Rend. Circ. Mat. Palermo (2) **72** (2023), no. 5, 3223–3253.

[RW20] E. A. Romano and J. A. Wiśniewski, *Adjunction for varieties with a* $\mathbb{C}^*$*-action*, Transform. Groups **27** (2022), no. 4, 1431–1473.

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